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**LETTER TO THE EDITOR**

# Fourier–Gauss transforms of the Askey–Wilson polynomials

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**Abstract.** We discuss classical Fourier–Gauss integral transforms for a five-parameter family of the Askey–Wilson polynomials $p_n(x; a, b, c, d|q)$.

The goal of the present paper is to study the Fourier–Gauss transformation properties of a family of the five-parameter Askey–Wilson polynomials [1]

$$p_n(x; a, b, c, d|q) := a^{-n}(ab, ac, ad; q)_n {}_4\phi_3 \left[ \begin{matrix} q^{-n}, abcdq^{n-1}, a\,\mathrm{e}^{\mathrm{i}\theta}, a\,\mathrm{e}^{-\mathrm{i}\theta} \\ ab, ac, ad, \end{matrix} ; q, q \right]$$
$$= a^{-n}\,(ab, ac, ad; q)_n \sum_{k=0}^{n} \frac{(q^{-n}, abcdq^{n-1}, a\,\mathrm{e}^{\mathrm{i}\theta}, a\,\mathrm{e}^{-\mathrm{i}\theta}; q)_k}{(ab, ac, ad, q; q)_k} q^k \qquad (1)$$

in the variable $x = \cos\theta$, $0 \leqslant \theta \leqslant \pi$. Throughout this paper we will employ the standard notations of $q$-analysis [2, 3], in particular

$$(a; q)_k = \prod_{j=0}^{k-1}(1 - aq^j) \qquad (a_1, \dots, a_n; q)_k = \prod_{j=1}^{n}(a_j; q)_k. \qquad (2)$$

The Askey–Wilson polynomials (1) are symmetric with respect to the four parameters $a, b, c, d$ and

$$p_n(-x; a, b, c, d|q) = (-1)^n p_n(x; -a, -b, -c, -d|q). \qquad (3)$$

For the values $0 < |q| < 1$ of the parameter $q$ they are orthogonal over the finite interval $-1 \leqslant x \leqslant 1$ with respect to the continuous measure

$$w_{AW}(x; a, b, c, d|q)\mathrm{d}x = 2q^{-1/8}e_q(q)\vartheta_1(\theta, q^{1/2}) \prod_{v=a,b,c,d} e_q(v\,\mathrm{e}^{\mathrm{i}\theta})e_q(v\,\mathrm{e}^{-\mathrm{i}\theta}) \sin\theta\,\mathrm{d}\theta \qquad (4)$$

where $\vartheta_1(z, q)$ is the theta-function and the $q$-exponential function $e_q(z)$ is given by

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = (z; q)_\infty^{-1}. \qquad (5)$$

Later on it has become clear that the modular and periodic properties of the theta-function, which enters the measure (4), allow one to introduce another continuous orthogonality

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relation for the Askey–Wilson polynomials $p_n(\sin\kappa s; a, b, c, d|q)$, $q = \exp(-2\kappa^2)$, over the full real line $-\infty < s < \infty$. The Ramanujan-type measure with an infinite support

$$w(\sin\kappa s; a, b, c, d|q)\,\mathrm{d}\sin\kappa s = \prod_{v=a,b,c,d} e_q(\mathrm{i}v\,\mathrm{e}^{-\mathrm{i}\kappa s})\,e_q(-\mathrm{i}v\,\mathrm{e}^{\mathrm{i}\kappa s})\,\mathrm{e}^{-s^2}\cos\kappa s\,\mathrm{d}s \tag{6}$$

thus obtained from (4) has an advantage in that it admits the transformation $q \to q^{-1}$ [4–6].

Once the Ramanujan-type continuous measure of orthogonality (6) was established, it has been realized that the classical Fourier–Gauss transform might relate the Askey–Wilson polynomials (1) with different values of the parameter $q$. This proved to be true first for the $q$-Hermite polynomials [7], which are a particular case of the Askey–Wilson polynomials $p_n(x; a, b, c, d|q)$ with vanishing parameters $a, b, c, d$. The corresponding Fourier–Gauss transform has the form

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}sr-s^2/r}H_n(\sin\kappa s|q)\,\mathrm{d}s = \mathrm{i}^n q^{n^2/4}h_n(\sinh\kappa r|q)\,\mathrm{e}^{-r^2/2} \tag{7}$$

where $q = \exp(-2\kappa^2)$ and $h_n(x|q) := \mathrm{i}^{-n}H_n(\mathrm{i}x|q^{-1})$ are the $q^{-1}$-Hermite polynomials [8]. Then the big $q$-Hermite polynomials $H_n(x; a|q) := p_n(x; a, 0, 0, 0|q)$ and Al-Salam–Chihara polynomials $p_n(x; a, b) := p_n(x; a, b, 0, 0|q)$ have been shown to have simple behaviour with respect to the classical Fourier–Gauss transform [9, 10].

The derivation of the Fourier–Gauss transforms in [9] and [10] is based upon combining the integral transform (7) with the expansions (or connection coefficients formulae) for the big $q$-Hermite and the Al-Salam–Chihara polynomials in terms of the $q$-Hermite polynomials. It turns out that one can employ the same idea for the general case of the Askey–Wilson polynomials $p_n(x; a, b, c, d|q)$ with non-zero values of the parameters $a, b, c, d$.

We begin with the relation

$$p_n(x; a, b, c, d|q) = \sum_{k=0}^{n} C_{n,k}\,p_k(x; a, \beta, \gamma, \delta|q) \tag{8}$$

between the Askey–Wilson polynomials, which depend on two sets of the parameters $a, b, c, d$ and $a, \beta, \gamma, \delta$. The connection coefficients in (8)

$$C_{n,k} = C_{n,k}(a, b, c, d; a, \beta, \gamma, \delta|q) := q^{k(k-n)}\begin{bmatrix} n \\ k \end{bmatrix}_q a^{k-n}\frac{(ab, ac, ad; q)_n(abcdq^{n-1}; q)_k}{(ab, ac, ad, a\beta\gamma\delta q^{k-1}; q)_k}$$
$$\times {}_5\phi_4\left[\begin{matrix} q^{k-n}, abcdq^{n+k-1}, a\beta q^k, a\gamma q^k, a\delta q^k \\ a\beta\gamma\delta q^{2k}, abq^k, acq^k, adq^k \end{matrix}; q, q\right] \tag{9}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the $q$-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k\,(q; q)_{n-k}} \tag{10}$$

have been derived in [1] by using the finite-interval orthogonality relation for the Askey–Wilson polynomials (see [2]). The particular case of (8) with $\beta = \gamma = \delta = 0$ gives the relation

$$p_n(x; a, b, c, d|q) = \sum_{k=0}^{n} A_{n,k}(q)H_k(x; a|q) \tag{11}$$

between the Askey–Wilson $p_n(x; a, b, c, d|q)$ and the big $q$-Hermite $H_k(x; a|q)$ polynomials. As follows from (9), the constant $A_{n,k}(q)$ in (11) is equal to

$$\begin{aligned} A_{nk}(q) &:= C_{n,k}(a, b, c, d; a, 0, 0, 0|q) \\ &= q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{k-n} (abcdq^{n-1}; q)_k (abq^k, acq^k, adq^k; q)_{n-k} \\ &\quad \times {}_4\phi_3 \left[ \begin{matrix} q^{k-n}, abcdq^{n+k-1}, 0, 0 \\ abq^k, acq^k, adq^k \end{matrix} ; q, q \right]. \end{aligned} \tag{12}$$

Note that the inverse expansion with respect to (11)

$$H_n(x; a|q) = \sum_{k=0}^{n} A_{n,k}^{-1}(q) p_k(x; a, b, c, d|q) \tag{13}$$

is again a particular case of (9), but with the vanishing parameters $b, c, d$. The explicit form of the constant $A_{n,k}^{-1}(q)$ is

$$\begin{aligned} A_{n,k}^{-1}(q) &:= C_{n,k}(a, 0, 0, 0; a, b, c, d|q) \\ &= q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{a^{k-n}}{(abcdq^{k-1}; q)_k} {}_4\phi_3 \left[ \begin{matrix} q^{k-n}, abq^k, acq^k, adq^k \\ abcdq^{2k}, 0, 0 \end{matrix} ; q, q \right]. \end{aligned} \tag{14}$$

The next step is to combine the expansion (11) with the classical Fourier–Gauss transform for the continuous big $q$-Hermite polynomials, derived in [9]. It has two alternative forms

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isr - s^2/2} H_n(\sin \kappa s; a|q)\, ds = \begin{cases} i^n q^{n^2/4} e^{-r^2/2} \sum_{k=0}^{n} q^{3k^2/4-(n+1)k/2} \begin{bmatrix} n \\ k \end{bmatrix}_q \\ \qquad \times (ia)^k h_{n-k}(\sinh \kappa r|q) & (15a) \\ i^n q^{n^2/4} e^{-r^2/2} \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q c_{k,n}(q) \\ \qquad \times (-ia)^k h_{n-k}(\sinh \kappa r; a|q) & (15b) \end{cases}$$

where the constant $c_{k,n}(q)$ is equal to

$$c_{k,n}(q) = \sum_{j=0}^{k} \frac{(q^{-k}; q)_j}{(q; q)_j} q^{(n+j/2)j/2}. \tag{16}$$

Multiplying thus both sides of (11) by the factor $(2\pi)^{-1} \exp(isr - s^2/2)$ and integrating them over the variable $s$ within infinite limits by using first (15$a$) gives

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isr - s^2/2} p_n(\sin \kappa s; a, b, c, d|q)\, ds = e^{-r^2/2} \sum_{k=0}^{n} B_{n,k}(q) h_n(\sinh \kappa r|q). \tag{17a}$$

Here the constant $B_{n,k}(q)$ is equal to

$$B_{n,k}(q) = i^k q^{k^2/4} \sum_{j=0}^{n-k} q^{j(j-1)/2} \begin{bmatrix} k+j \\ j \end{bmatrix}_q (-a)^j A_{n,k+j}(q). \tag{18}$$

In the particular case when the parameters $c = d = 0$, the constant $A_{n,k}(q)$ reduces to

$$A_{n,k}(q) = q^{(n-k)(n-k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q (-b)^{n-k} \qquad c = d = 0. \tag{19}$$

To verify (19), one needs to use in (12) a special case of the Chu–Vandermonde $q$-sum for ${}_2\phi_1(q^{-n}, b; c; q, q)$ with a vanishing parameter $b$, that is

$$ {}_2\phi_1(q^{-n}, 0; c; q, q) = q^{n(n-1)/2} \frac{(-c)^n}{(c; q)_n}. \tag{20} $$

Consequently, for $c = d = 0$ the Fourier–Gauss transform (17) reduces to that for the Al-Salam–Chihara polynomials

$$ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sr - s^2/2} p_n(\sin\kappa s; a, b|q)\, \mathrm{d}s = \mathrm{i}^n q^{n^2/4}\, \mathrm{e}^{-r^2/2} \sum_{k=0}^{n} q^{3k^2/4-(n+1)k/2} \begin{bmatrix} n \\ k \end{bmatrix}_q (\mathrm{i}a)^k s_k(b/a; q) h_{n-k}(\sinh\kappa r|q) \tag{21} $$

derived in [10]. The Stielties–Wigert polynomials $s_k(z; q)$ in (21) are defined [2, 11] as

$$ s_n(z; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} z^k. \tag{22} $$

In a manner similar to the derivation of (17), from (11) and (15$b$) it follows that one can alternatively represent (17$a$) as

$$ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sr - s^2/2} p_n(\sin\kappa s; a, b, c, d|q)\, \mathrm{d}s = \mathrm{e}^{-r^2/2} \sum_{k=0}^{n} D_{n,k}(q) h_k(\sinh\kappa r; a|q) \tag{17$b$} $$

where the constant $D_{n,k}(q)$ is equal to

$$ D_{n,k}(q) = \mathrm{i}^k \sum_{j=0}^{n-k} q^{(k-j)^2/4} \begin{bmatrix} k+j \\ j \end{bmatrix}_q a^j c_{j,k+j}(q) A_{n,k+j}(q) \tag{23} $$

whereas $A_{n,k}(q)$ and $c_{n,k}(q)$ are given by (12) and (16), respectively. The Fourier–Gauss transform (17$b$) enables one to represent its right-hand side in terms of the Askey–Wilson $q^{-1}$-polynomials [5, 6]

$$ \tilde{p}_n(x; a, b, c, d|q) := \mathrm{i}^{-n} p_n(\mathrm{i}x; a, b, c, d|q^{-1}). \tag{24} $$

Indeed, from (13) and (24) it follows that

$$ h_n(x; a|q) = \sum_{k=0}^{n} \mathrm{i}^{k-n} A^{-1}_{n,k}(q^{-1}) \tilde{p}_k(x; a, b, c, d|q). \tag{25} $$

Now substituting (25) in the right-hand side of (17$b$) gives

$$ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sr - s^2/2} p_n(\sin\kappa s; a, b, c, d|q)\, \mathrm{d}s = \mathrm{e}^{-r^2/2} \sum_{k=0}^{n} E_{n,k}(q) \tilde{p}_k(\sinh\kappa r; a, b, c, d|q) \tag{17$c$} $$

where the constant

$$ E_{n,k}(q) = \sum_{j=0}^{n-k} \mathrm{i}^{-j} D_{n,k+j}(q) A^{-1}_{k+j,k}(q^{-1}). \tag{26} $$

Note that an explicit form of the constant $A^{-1}_{n,k}(q^{-1})$, which enters the relations (25) and (26), is readily obtained from (14) upon using the inversion formula

$$ (a; q^{-1})_n = q^{-n(n-1)/2} (-a)^n (a^{-1}; q)_n \tag{27} $$

with respect to the transformation $q \to q^{-1}$. In particular, when $a = q^{-1}$ from (27) follows that the $q$-binomial coefficient (10) enjoys the property

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q . \tag{28}$$

To derive another pair of Fourier–Gauss transforms for the Askey–Wilson polynomials (1) one may start with the inverse expansion with respect to (25). In (11) change $x \to \mathrm{i}x$ and transform $q \to q^{-1}$. Taking into account the definition (24), this results in

$$\tilde{p}_n(x; a, b, c, d|q) = \sum_{k=0}^{n} \mathrm{i}^{k-n} A_{n,k}(q^{-1}) h_k(x; a|q). \tag{29}$$

The relation (29) can be combined with the Fourier–Gauss transforms for the big $q^{-1}$-Hermit polynomials [9]

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sr - s^2/2} h_n(\sinh \kappa s; a|q)\, \mathrm{d}s$$

$$= \mathrm{i}^{-n} q^{-n^2/4}\, \mathrm{e}^{-r^2/2} \sum_{k=0}^{n} q^{k^2/4+(1-n)k/2} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^k H_{n-k}(\sin \kappa r|q) \tag{30a}$$

$$= \mathrm{i}^{-n} q^{-n^2/4}\, \mathrm{e}^{-r^2/2} \sum_{k=0}^{n} c_{k,n}(q^{-1}) \begin{bmatrix} n \\ k \end{bmatrix}_q a^k H_{n-k}(\sin \kappa r; a|q) \tag{30b}$$

which are related to (15*a*) and (15*b*), respectively, by a replacement of the base $q \to q^{-1}$ (i.e. $\kappa \to \mathrm{i}\kappa$). Multiplying thus both sides of (29) for $x = \sinh \kappa x$ by the factor $(2\pi)^{-1/2} \exp(-\mathrm{i}sr - s^2/2)$ and integrating them over the variable $s$ within infinite limits by using (30*a*) or (30*b*) gives

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}sr - s^2/2} \tilde{p}_n(\sinh \kappa s; a, b, c, d|q)\, \mathrm{d}s = \mathrm{e}^{-r^2/2} \sum_{k=0}^{n} M_{n,k}(q) H_k(\sin \kappa r|q) \tag{31a}$$

$$= \mathrm{e}^{-r^2/2} \sum_{k=0}^{n} N_{n,k}(q) H_k(\sin \kappa r; a|q) \tag{31b}$$

where the constants $M_{n,k}(q)$ and $N_{n,k}(q)$ are given by

$$M_{n,k}(q) = \mathrm{i}^{-n} q^{-k^2/4} \sum_{j=0}^{n-k} q^{-j(j-1)/2-kj} \begin{bmatrix} k+j \\ j \end{bmatrix}_q (-a)^j A_{n,k+j}(q^{-1}) \tag{32a}$$

$$N_{n,k}(q) = \mathrm{i}^{-n} \sum_{j=0}^{n-k} q^{-(k+j)^2/4} \begin{bmatrix} k+j \\ j \end{bmatrix}_q a^j A_{n,k+j}(q^{-1}) c_{j,k+j}(q^{-1}). \tag{32b}$$

It remains only to employ the expansion (13) in the right-hand side of (31*b*) to obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}sr - s^2/2} \tilde{p}_n(\sinh \kappa s; a, b, c, d|q)\, \mathrm{d}s$$

$$= \mathrm{e}^{-r^2/2} \sum_{k=0}^{n} R_{n,k}(q) p_k(\sin \kappa r; a, b, c, d|q). \tag{31c}$$

Here, the constant $R_{n,k}(q)$ is equal to

$$R_{n,k}(q) = E^{-1}_{n,k}(q) = \sum_{j=0}^{n-k} N_{n,k+j}(q) A^{-1}_{k+j,k}(q). \tag{33}$$

The relations (17) and (31) thus provide explicit forms of the Fourier–Gauss transforms for a hierarchy of the five-parameter Askey–Wilson polynomials (1), defined in terms of the basic hypergeometric series ${}_4\phi_3$. When two or three of the parameters $a, b, c, d$ vanish, they coincide with the Fourier–Gauss transforms for the Al-Salam–Chihara [10], and the big $q$-Hermite [9] polynomials, respectively. In principle, one can use the same technique for finding Fourier–Gauss transforms of more general families of basic hypergeometric polynomials (for instance, at the ${}_8\phi_7$-level), but their explicit forms become more and more complicated as a number of the parameters increases.

Another interesting research direction is to find Fourier–Gauss transforms for the ${}_4\phi_3$ biorthogonal rational functions of Al-Salam and Ismail [12, 13]. They depend on the same number of parameters as the Askey–Wilson family (1). The lower level of the Al-Salam–Ismail rational functions (that is, their particular case with the four vanishing parameters $a, b, c$ and $d$) corresponds to the Rogers–Szegö polynomials [14–16], which are related to the Stieltjest–Wigert polynomials [2] through the classical Fourier–Gauss transform [**?**]. It is of interest to find Fourier–Gauss transforms for the next levels of the Al-Salam–Ismail family.

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